Partial Differential Equations of Second Order:

If a pde contains at least one of the second order partial derivatives, r, s or t but not of higher order, then it is a second order pde.

Origin:

Let $z = f(u) + g(v) + w \longrightarrow (1)$

where u, v and w are functions of x and y.

Differentiating (1) partially w.r.t. x and y:

$$p = f'(u)\,u_x + g'(v)\,v_x + w_x \longrightarrow (2)$$

$$q = f'(u)\,u_y + g'(v)\,v_y + w_y \longrightarrow (3)$$

where $u_x = \frac{\partial u}{\partial x}$, $v_x = \frac{\partial v}{\partial x}$, $w_x = \frac{\partial w}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, $v_y = \frac{\partial v}{\partial y}$ and $w_y = \frac{\partial w}{\partial y}$.

$$r = f''(u)\,u_x^2 + f'(u)\,u_{xx} + g''(v)\,v_x^2 + g'(v)\,v_{xx} + w_{xx} \longrightarrow (4)$$

$$s = f''(u)\,u_y u_x + f'(u)\,u_{xy} + g''(v)\,v_y v_x + g'(v)\,v_{xy} + w_{xy} \longrightarrow (5)$$

$$t = f''(u)\,u_y^2 + f'(u)\,u_{yy} + g''(v)\,v_y^2 + g'(v)\,v_{yy} + w_{yy} \longrightarrow (6)$$

Eliminating f', f'', g', g'' from (2), (3), (4), (5) & (6):

$$\begin{vmatrix} p - w_x & u_x & v_x & 0 & 0 \\ q - w_y & u_y & v_y & 0 & 0 \\ r - w_{xx} & u_{xx} & v_{xx} & u_x^2 & v_x^2 \\ s - w_{xy} & u_{xy} & v_{xy} & u_y u_x & v_y v_x \\ t - w_{yy} & u_{yy} & v_{yy} & u_y^2 & v_y^2 \end{vmatrix} = 0$$

$$\Rightarrow Rr + Ss + Tt + Pp + Qq = W \longrightarrow (7)$$

where R, S, T, P, Q & W are functions of x & y.

Canonical Forms:

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \longrightarrow (8)$$

Let $u = u(x, y)$, $v = v(x, y)$

$$p = \frac{\partial z}{\partial u}\,u_x + \frac{\partial z}{\partial v}\,v_x = z_u u_x + z_v v_x$$

$$q = z_u u_y + z_v v_y$$

$r = (z_{uu} u_x + z_{uv} v_x) u_x + z_u u_{xx}$
$+ (z_{uv} u_x + z_{vv} v_x) v_x + z_v v_{xx}$
$= z_{uu} u_x^2 + 2 z_{uv} u_x v_x + z_{vv} v_x^2 + z_u u_{xx} + z_v v_{xx}.$

$s = (z_{uu} u_y + z_{uv} v_y) u_x + z_u u_{xy}$
$+ (z_{uv} u_y + z_{vv} v_y) v_x + z_v v_{xy}$
$= z_{uu} u_y u_x + z_{uv} (v_y u_x + u_y v_x) + z_{vv} v_y v_x$
$+ z_u u_{xy} + z_v v_{xy}$

$t = (z_{uu} u_y + z_{uv} v_y) u_y + z_u u_{yy}$
$+ (z_{uv} u_y + z_{vv} v_y) v_y + z_v v_{yy}$
$= z_{uu} u_y^2 + 2 z_{uv} u_y v_y + z_{vv} v_y^2 + z_u u_{yy} + z_v v_{yy}.$

(8) $\Rightarrow A z_{uu} + 2B z_{uv} + C z_{uv} + F(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}) = 0 \longrightarrow$ (9)
where
$A = R u_x^2 + S u_x u_y + T u_y^2$
$B = R u_x v_x + \frac{1}{2} S (u_x v_y + u_y v_x) + T u_y v_y$
$C = R v_x^2 + S v_x v_y + T v_y^2$

If $u_x = \lambda u_y$ and $v_x = \lambda v_y$, then
(9) $\Rightarrow (R\lambda^2 + S\lambda + T)(u_y^2 z_{uu} + 2u_y v_y z_{uv} + v_y^2 z_{vv}) = 0$

$\Rightarrow R\lambda^2 + S\lambda + T = 0$
Case I : $S^2 - 4RT > 0$
There are two distinct real roots λ_1 & λ_2.
$\Rightarrow u_x = \lambda_1 u_y$ and $v_x = \lambda_2 v_y$.
So $A = (R\lambda_1^2 + S\lambda_1 + T) u_y^2 = 0$
& $C = (R\lambda_2^2 + S\lambda_2 + T) v_y^2 = 0$

Now $u_x - \lambda_1 u_y = 0$
$\Rightarrow \frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{du}{0}$
$\cancel{\Rightarrow y + \lambda_1 x = c \text{ and } u = f_1(y + \lambda_1 x)}$
$f_1(x, y) = c$ and $u = c = f_1(x, y)$

$v_x - \lambda_2 v_y = 0$
$\Rightarrow \frac{dx}{1} = \frac{dy}{-\lambda_2} = \frac{dv}{0}$
$\cancel{\Rightarrow y - \lambda_2 x = a \text{ and } v = f_2(y - \lambda_2 x)}$
$f_2(x, y) = c'$
and $v = c' = f_2(x, y)$.

(9) $\Rightarrow$ $2B z_{uv} + F(u, v, z, z_u, z_v) = 0$

$$AC - B^2 = 0 - B^2 = \frac{1}{4}(4RT - S^2)(u_x v_y - u_y v_x)^2$$

$$\Rightarrow B^2 = \frac{1}{4}(S^2 - 4RT)(u_x v_y - u_y v_x)^2 \geq 0$$

$$\Rightarrow \frac{\partial^2 z}{\partial u \partial v} = z_{uv} = \phi(u, v, z, z_u, z_v) \rightarrow (10)$$

$\rightarrow$ canonical form

Equation (8) is hyperbolic if $S^2 - 4RT > 0$

Case II : $S^2 - 4RT = 0$

We have only one root $\lambda = \lambda_1$. As before.

$u = f_1(x, y)$, $f_1(x, y) = c_1$

We chose $v = v(x, y)$ such that u and v are independent functions.

$$A = (R\lambda_1^2 + S\lambda_1 + T) u_y^2 = 0$$

$$\therefore AC - B^2 = 0 - B^2 = \frac{1}{4}(4RT - S^2)(u_x v_y - u_y v_x)^2 = 0$$

$$\Rightarrow B = 0.$$

So $z_{vv} = \frac{\partial^2 z}{\partial v^2} = \phi(u, v, z, z_u, z_v) \rightarrow (11)$

$\longrightarrow$ canonical form

(8) is called parabolic if $S^2 - 4RT = 0$

Case III $S^2 - 4RT < 0$.

$\lambda = \lambda_1, \lambda_2$ are distinct complex roots.

So u and v are complex conjugates

Let $u = \alpha + i\beta$, $v = \alpha - i\beta$

$\alpha = \frac{u+v}{2}$ $\qquad$ $\beta = -\frac{(u-v)}{2} i$

$$z_u = z_\alpha \alpha_u + z_\beta \beta_u = \frac{1}{2}(z_\alpha - i z_\beta)$$

$$z_v = z_\alpha \alpha_v + z_\beta \beta_v = \frac{1}{2}(z_\alpha + i z_\beta)$$

$$z_{uv} = (z_{\alpha\alpha}\alpha_v + z_{\alpha\beta}\beta_v)\alpha_u + z_\alpha \alpha_{uv} + (z_{\alpha\beta}\alpha_v + z_{\beta\beta}\beta_v)\beta_u + z_\beta \beta_{uv}$$

$$= z_{\alpha\alpha}\frac{1}{4} + z_{\alpha\beta}\left(\frac{i}{4}-\frac{i}{4}\right) + z_{\beta\beta}\frac{i}{2}\left(\frac{-i}{2}\right)$$
$$= \frac{1}{4}\left(z_{\alpha\alpha}+z_{\beta\beta}\right)$$

(10) $\Rightarrow$ $$z_{\alpha\alpha}+z_{\beta\beta} = \psi_1(\alpha,\beta,z,z_\alpha,z_\beta) \rightarrow (12)$$
$\hookrightarrow$ canonical form

(8) is called <u>elliptic</u> if $S^2-4RT<0$

Ex Solve $t - xq = x^2$

$t = \frac{\partial q}{\partial y} \Rightarrow \frac{\partial q}{\partial y} - xq = x^2$

This is a linear equation in q with integrating factor $e^{\int -x\,dy} = e^{-xy}$

$$\Rightarrow e^{-xy}\frac{\partial q}{\partial y} - xqe^{-xy} = x^2e^{-xy}$$

$$\Rightarrow \frac{\partial}{\partial y}\left(e^{-xy}q\right) = x^2e^{-xy}$$

$$\Rightarrow e^{-xy}q = -xe^{-xy} + A$$ where A is constant w.r.t. y

So we can take $A = \phi(x)$

$$\Rightarrow q = \frac{\partial z}{\partial y} = -x + \phi(x)e^{xy}$$

$$\Rightarrow z = -xy + \int\phi(x)e^{xy}\,dy + \psi_1(x)$$
$$= -xy + \frac{\phi(x)}{x}e^{xy} + \psi_1(x)$$
$$= -xy + \psi_2(x)e^{xy} + \psi_1(x).$$

Ex $ys + p = \cos(x+y) - y\sin(x+y)$. $s = \frac{\partial p}{\partial y}$

$$y\frac{\partial p}{\partial y} + p = \cos(x+y) - y\sin(x+y)$$

$$\Rightarrow \frac{\partial}{\partial y}(yp) = \cos(x+y) - y\sin(x+y)$$

$$\Rightarrow yp = \sin(x+y) + y\cos(x+y) - \sin(x+y) + \phi(x)$$
$$= y\cos(x+y) + \phi(x)$$

$$\Rightarrow \frac{\partial z}{\partial x} = \cos(x+y) + \frac{1}{y}\phi(x)$$

$$\Rightarrow z = \sin(x+y) + \frac{1}{y}\int\phi(x)\,dx + \psi_1(y)$$
$$= \sin(x+y) + \frac{1}{y}\phi_1(x) + \psi_1(y)$$

Ex Find the surface of revolution that touches the circle $z=0=x^2+y^2-1$ and satisfies $s=8xy$

$s=8xy \Rightarrow \frac{\partial p}{\partial y}=8xy$

$\Rightarrow p=4xy^2+\phi(x)$

$\Rightarrow z=2x^2y^2+\int\phi(x)\,dx+\psi_1(y)$

$=2x^2y^2+\psi_2(x)+\psi_1(y)$

$z=0 \Rightarrow \psi_2(x)+\psi_1(y)=-2x^2y^2$

$x^2+y^2=1 \Rightarrow z=2x^2(1-x^2)-2x^2y^2$

$=2x^2(1-x^2-y^2)$

Ex Reduce the equation $y^2r=t$ to the canonical form

$y^2r-t=0$

$\Rightarrow R=y^2 \quad S=0 \quad T=-1$

$\Rightarrow S^2-4RT=4y^2>0$

So the equation is hyperbolic in nature.

$R\lambda^2+S\lambda+T=y^2\lambda^2-1=0 \Rightarrow \lambda_1=\frac{1}{y}, \lambda_2=\frac{-1}{y}$

$\frac{\partial u}{\partial x}=\frac{1}{y}\frac{\partial u}{\partial y} \Rightarrow y\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}=0$

$\Rightarrow \frac{dx}{y}=\frac{dy}{-1}=\frac{du}{0} \Rightarrow x+\frac{y^2}{2}=c=u$

$\frac{\partial v}{\partial x}=-\frac{1}{y}\frac{\partial v}{\partial y} \Rightarrow y\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}=0$

$\Rightarrow \frac{dx}{y}=\frac{dy}{1}=\frac{dv}{0} \Rightarrow x-\frac{y^2}{2}=c_1=v$

$\therefore x=\frac{u+v}{2}$ and $y^2=u-v$

Now $p=z_u u_x+z_v v_x=z_u+z_v$

$q=z_u u_y+z_v v_y=y(z_u-z_v)$

$r=z_{uu}u_x+z_{uv}v_x+z_{uv}u_x+z_{vv}v_x$

$=z_{uu}+2z_{uv}+z_{vv}$

$t=z_u-z_v+y(z_{uu}u_y+z_{uv}v_y-z_{uv}u_y-z_{vv}v_y)$

$=z_u-z_v+y^2(z_{uu}-2z_{uv}+z_{vv})$

$\therefore\; y^2 r - t = (u-v)(z_{uu} + 2z_{uv} + z_{vv})$

$\quad - z_u + z_v - (u-v)(z_{uu} - 2z_{uv} + z_{vv})$

$= 4(u-v) z_{uv} - z_u + z_v = 0$

$$\Rightarrow z_{uv} = \frac{z_u - z_v}{4(u-v)}$$

<u>Ex</u> Reduce to the canonical form and solve:

$y^2 r - 2xy\, s + x^2 t = \frac{y^2}{x} p + \frac{x^2}{y} q \quad \to (1)$

$R = y^2 \qquad S = -2xy \qquad T = x^2$

$\Rightarrow \quad S^2 - 4RT = 4x^2y^2 - 4x^2y^2 = 0$

The equation is parabolic in nature

$R\lambda^2 + S\lambda + T = y^2\lambda^2 - 2xy\lambda + x^2 = (y\lambda - x)^2 = 0$

$\lambda = \frac{x}{y}$

$u_x = \frac{x}{y} u_y \quad \Rightarrow \quad y u_x - x u_y = 0$

$\Rightarrow \quad \frac{dx}{y} = \frac{-dy}{+x} = \frac{du}{0}$

$\Rightarrow \quad x^2 + y^2 = c = u$

We can take $v = x^2 - y^2$ so that $x^2 = \frac{u+v}{2}$

and $y^2 = \frac{u-v}{2}$

$p = z_u u_x + z_v v_x = 2x(z_u + z_v)$

$q = z_u u_y + z_v v_y = 2y\, (z_u - z_v)$

$r = \frac{\partial p}{\partial x} = 2(z_u + z_v) + 2x(z_{uu} u_x + z_{uv} v_x$

$\quad + z_{uv} u_x + z_{vv} v_x)$

$= 2(z_u + z_v) + 4x^2 (z_{uu} + z_{vv} + 2z_{uv})$

$s = \frac{\partial p}{\partial y} = 2x(z_{uu} u_y + z_{uv} v_y + z_{uv} u_y + z_{vv} v_y)$

$= 4xy(z_{uu} - z_{vv})$

$t = \frac{\partial q}{\partial y} = 2(z_u - z_v) + 2y(z_{uu} u_y + z_{uv} v_y - z_{uv} u_y$

$\quad - z_{vv} v_y)$

$= 2(z_u - z_v) + 4y^2 (z_{uu} - 2z_{uv} + z_{vv})$

$\therefore (1) \Rightarrow \left(\frac{u-v}{2}\right)\left\{ 2z_u + 2z_v + (u+v)(z_{uu} + z_{vv} + 2z_{uv}) \right\}$

$- 2(u^2 - v^2)(z_{uu} - z_{vv}) + (u+v)\{ z_u - z_v + (u-v)(z_{uu} - 2z_{uv}$

$\quad + z_{vv})$

$= (u-v)(z_u + z_v) + (u+v)(z_u - z_v)$

$\Rightarrow (u^2-v^2)(z_{uu} + z_{vv} + 2z_{uv} - 2z_{uu} + 2z_{uv}$
$+ z_{uu} - 2z_{uv} + z_{vv}) = 0$

$\Rightarrow z_{vv} = 0$

$\Rightarrow z_v = f(u)$

$\Rightarrow z = v f(u) + g(u)$

$= (x^2-y^2) f(x^2+y^2) + g(x^2+y^2)$

Ex $r + x^2 t = 0$ Reduce this equation to canonical form.

$R = 1 \quad T = x^2 \quad S = 0$

$S^2 - 4RT = -4x^2 < 0$

The equation is elliptic in nature

$R\lambda^2 + S\lambda + T = \lambda^2 + x^2 = 0 \Rightarrow \lambda = \pm ix$

$\frac{\partial u}{\partial x} = ix \frac{\partial u}{\partial y} \Rightarrow \frac{dx}{1} = \frac{dy}{-ix} = \frac{du}{0}$

$\Rightarrow i\frac{x^2}{2} + y = u = \alpha + i\beta$

$\frac{\partial v}{\partial x} = -ix \frac{\partial v}{\partial y} \Rightarrow v = -i\frac{x^2}{2} + y = \alpha - i\beta$

$\Rightarrow \alpha = y \quad \beta = \frac{x^2}{2}$

$p = z_\alpha \alpha_x + z_\beta \beta_x = z_\beta x$

$q = z_\alpha \alpha_y + z_\beta \beta_y = z_\alpha$

$r = \frac{\partial p}{\partial x} = z_\beta + x(z_{\alpha\beta}\alpha_x + z_{\beta\beta}\beta_x)$

$= z_\beta + x^2 z_{\beta\beta}$

$t = \frac{\partial q}{\partial y} = z_{\alpha\alpha}\alpha_y + z_{\alpha\beta}\beta_y$

$= z_{\alpha\alpha}$

$\therefore r + x^2 t = z_\beta + 2\beta z_{\beta\beta} + 2\beta z_{\alpha\alpha} = 0$

$\Rightarrow (z_{\alpha\alpha} + z_{\beta\beta}) = -\frac{1}{2\beta} z_\beta$

$\leftrightarrow$